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On the topology of the Lü attractor and related systems

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Abstract
We use well-established methods of knot theory to study the topological structure of the set of periodic orbits of the Lü attractor. We show that, for a specific set of parameters, the Lü attractor is topologically different from the classical Lorenz attractor, whose dynamics is formed by a double cover of the simple horseshoe. This argues against the 'similarity' between the Lü and Lorenz attractors, claimed, for these parameter values, by some authors on the basis of non-topological observations. However, we show that the Lü system belongs to the Lorenz-like family, since by changing the values of the parameters, the behaviour of the system follows the behaviour of all members of this family. An attractor of the Lü kind with higher order symmetry is constructed and some remarks on the Chen attractor are also presented.

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1. Introduction

Lü and Chen introduced in [11] a simple 3-parameter family of ordinary differential equations (ODEs), nowadays called the Lü system, which exhibits chaotic behaviour. The equations of this system are

$$\begin{aligned}\dot{x} &= a(y-x)\\ \dot{y} &= -xz+cy\\ \dot{z} &= xy-bz\end{aligned} \tag{1}$$

where a, b, c are real parameters. In the same paper, Lü and Chen integrated these ODEs numerically, for fixed $a = 36$ and $b = 3$, varying only the third parameter. They observed that, when $12.7 < c < 17$, the attractor generated by (1) is pictorially 'similar' to the well-known

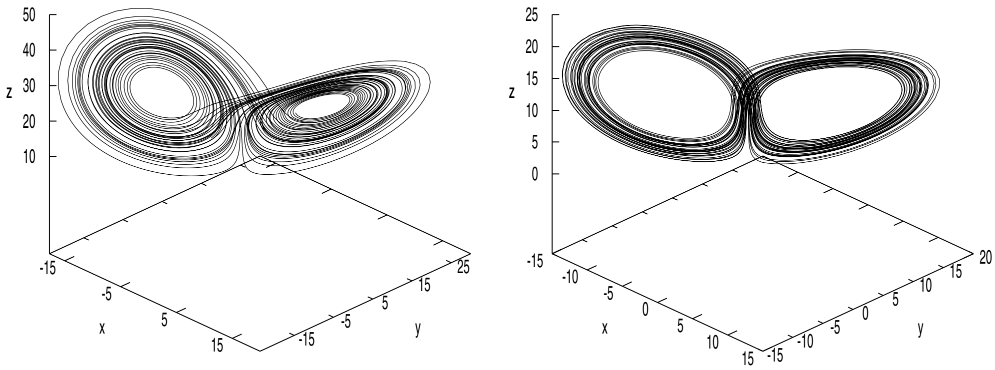

Figure 1. Lorenz attractor (left) for $(\sigma, \beta, \rho) = (10, 8/3, 28)$ and Lü attractor (right) for $(a, b, c) = (36, 3, 13)$.

Lorenz attractor [2]. They also noted that, for $18 < c < 22$, the attractor has an intermediate shape, while for $23 < c < 28.5$, it becomes visually 'similar' to Chen's attractor [10]. This result was also supported in [12] with the aid of Arnol'd's theory of unfolding of matrices [3]. Indeed, as figure 1 shows, the Lü attractor, for $(a, b, c) = (36, 3, 13)$, 'looks like' the Lorenz attractor, whose equations are

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z.
\end{aligned}$$

In the present paper, we use the method of topological analysis, introduced by Birman and Williams in [4] and further developed by Gilmore, Lefranc and Letellier *et al* [9, 5] (and references therein). We apply this method to the system obtained when the symmetry of the Lü system is modded out for $(a, b, c) = (36, 3, 13)$. As proved in [8], the symmetry-reduced Lorenz system exhibits straightforward (simple) horseshoe dynamics while the results of our analysis show that the basic mechanism underlying the attractor of system (1) is quite different. This allows us to argue that the Lü and Lorenz systems have, for specific parameter values, *different* topological properties and hence their 'similarity' under visual inspection is deceptive. The symmetry-induced system is then used to construct systems with higher order symmetries. Interestingly, the same procedure, when applied to the well-known Chen system indicates an even more complex topological structure, as we shall discuss below.

The similarities observed in these three systems can be explained only under a wider study of their global behaviour under variation of their parameters. Within such a study, we present here the perestroika that the Lü system undergoes, which classifies it as a member of the Lorenz-like family of dynamical systems [15].

This paper is organized as follows: in section 2 we construct a three-dimensional set of ODEs called the proto-Lü system, which is simpler than the Lü system and allows us to draw concrete conclusions about system (1). In section 3 topological invariants from knot theory are used to determine the topological structure of the proto-Lü system. Next, in section 4, we use the results obtained in [8] combined with our findings, to argue *against* the equivalence of the Lorenz and Lü systems for some choices of parameters. In section 5 we justify the use of the symmetry reduction method, by constructing a system with the same local properties as system (1), but with a higher order symmetry. Then, in section 6, we argue that the Chen attractor possesses an even higher degree of topological complexity, due to its considerably more complicated topological properties. Finally, the global properties of the Lü system, as well as the *perestroika* it undergoes are presented in section 7, allowing us to classify its dynamical complexity as similar to that of the Lorenz-like family. The last section contains the conclusions of our work.

2. The proto-Lü system

As is easily verified, system (1) remains invariant under an order two symmetry, expressed by the involution

$$\begin{aligned} &\mathbb{R}^3 \to \mathbb{R}^3 \\ &(x, y, z) \mapsto (-x, -y, z). \end{aligned}$$

This means that each orbit of the system is either self-symmetric or possesses a 'twin' orbit, which is symmetric with respect to the above involution.

It is more convenient, for computational as well as other reasons (discussed in section 5), to perform our analysis on the system obtained after removing this symmetry. To this end, we use the procedure described by Miranda and Stone in [7] to construct a vector field in $\mathbb{R}^3$ having no symmetry, but possessing the additional property that each phase curve of this field corresponds either to a pair of symmetric orbits or to a single self-symmetric orbit of system (1).

This is achieved through the map

$$\begin{aligned} &\pi : \mathbb{R}^3 \to \mathbb{R}^3 \\ &(x, y, z) \mapsto (x^2 - y^2, 2xy, z). \end{aligned} \tag{2}$$

Miranda and Stone proved that this map is the quotient map for the involution and is also a local diffeomorphism away of the z-axis. This means that the Lü system can be transformed to a new system, called the proto-Lü in accordance with [7]. This new system can be thought of as the projection of (1) on the orbit space of the above-mentioned symmetry. By that we mean that pairs of symmetric orbits of the Lü system (under the above symmetry) will be identified and correspond to a single orbit of the proto-Lü system, while self-symmetric orbits of the Lü system correspond to a single orbit of the new system.

To obtain this vector field, we only have to write the Lü system in new coordinates, i.e. we denote by N the quantity $x^2 + y^2 = \sqrt{u^2 + v^2}$, whence the equations of the proto-Lü system are

$$\begin{aligned} \dot{u} &= av - a(u + N) + vz - c(N - u) \\ \dot{v} &= a(N - u) - av - (u + N)z + cv \\ \dot{z} &= \tfrac{1}{2}v - bz. \end{aligned} \tag{3}$$

If we integrate system (3) numerically for parameter values $(a, b, c) = (36, 3, 13)$, the attractor generated is that shown in figure 2.

3. Topological analysis of the proto-Lü system

As expected, the proto-Lü system has two fixed points, namely $(0, 0, 0)$ and $(0, 2bc, c)$. The first one corresponds to the self-symmetric fixed point $(0, 0, 0)$, while the second corresponds to the pair of symmetric fixed points $(-\sqrt{bc}, -\sqrt{bc}, c)$, $(\sqrt{bc}, \sqrt{bc}, c)$ of the original system (1).

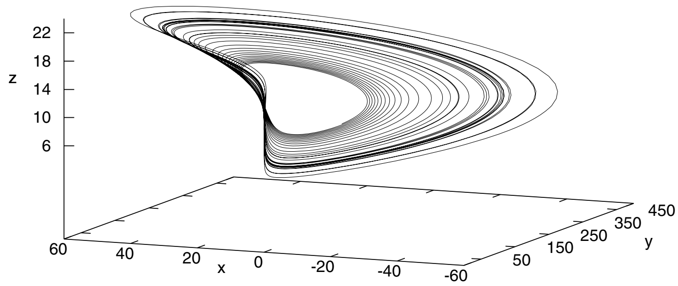

Figure 2. The proto-Lü attractor, where $(x, y, z) = (u, v, z)$.

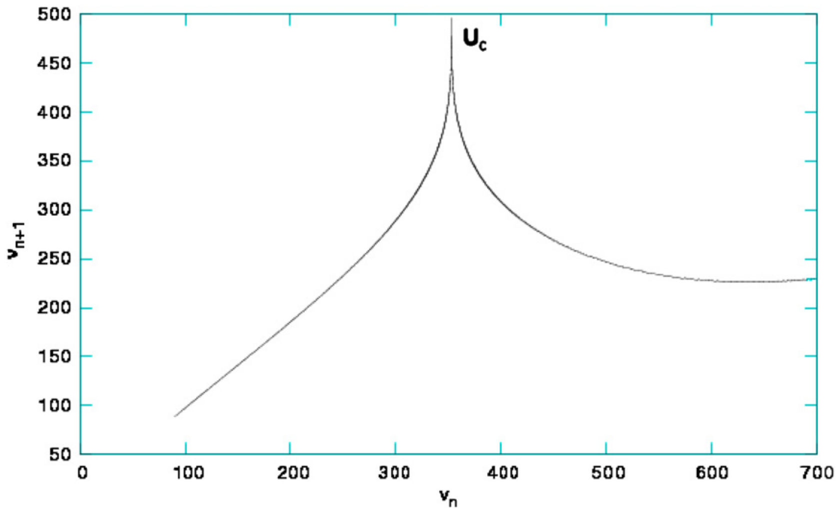

Figure 3. The first return map of the proto-Lü attractor with respect to the v variable.

We may think of the set of periodic orbits of an attractor as forming a kind of skeleton of the attractor [9]. Indeed, all other orbits contained in the attractor intertwine between the periodic orbits. Thus, determining the way that periodic orbits are *linked* [5], determines the structure of the attractor. The tools which allow us to study the topological structure of the periodic orbits come from knot theory, since any periodic orbit of a three-dimensional dynamical system can be considered as a knot.

A *template* of a three-dimensional dynamical system is a branched two-manifold introduced in [4], equipped with a semiflow. The topological structure of the periodic orbits of the semiflow is identical to the topological structure of the periodic orbits of the dynamical system corresponding to the template. In this section, we construct the template for the proto-Lü attractor, which allows us to draw conclusions about the structure of the set of periodic orbit of the proto-Lü system.

To do that, let us first define a Poincaré section for the system (3) as

$$P = \{(u, v, z) \in \mathbb{R}^3/z = c, \dot{z} > 0\} \tag{4}$$

and construct the first return map for this section with respect to the v variable. The result is shown in figure 3.

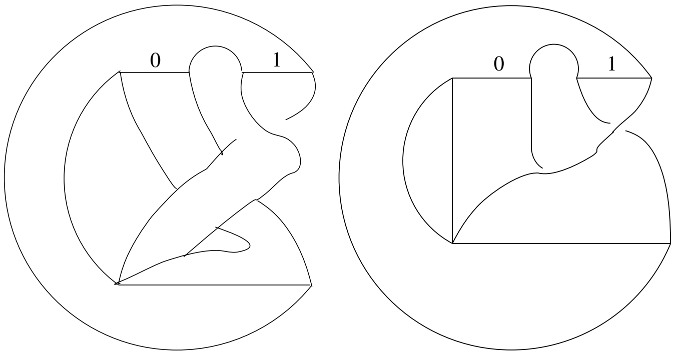

Figure 4. Templates of the proto-Lü (left) and proto-Lorenz systems (right).

Table 1. Topological invariants for the set of periodic orbits of system (3).

	0	01	001	0001	000 01
0	0				
01		+1	+1	+1	
001			+2	+2	
0001				+3	
000 01					+4

As seen in figure 3, this map has one critical point at $v_c = 353.081$, separating two branches, one increasing and one decreasing. This implies that the template describing the proto-Lü attractor consists also of two branches: one orientation preserving, which we shall name branch 0, and one orientation reversing, which we call branch 1. Looking at the figure more carefully, we observe that one more branch seems to exist in the first return map but this branch is not developed enough to be safely included in our analysis. Indeed, attempts to include this branch in our analysis showed that it does not change qualitatively the results described below.

Next, we compute all periodic orbits with period up to 5 located within the attractor. For the detection of periodic orbits, we used the methodology proposed in [17], and in particular the so-called differential evolution algorithm. To each periodic orbit we attribute symbolic names according to the following rule:

$$\mathrm{S}(v_n) = 0 \qquad \text{if} \quad v_n < v_c \qquad \text{and} \qquad \mathrm{S}(v_n) = 1 \qquad \text{if} \quad v_n > v_c.$$

Linking and self-linking numbers were computed for this set of periodic orbits and listed in table 1.

The first row and first column of table 1 contain the symbolic names of the periodic orbits extracted. The rest of the table presents the topological invariants of these orbits. Specifically, in the main diagonal we read the self-linking numbers of the orbits, while the off-diagonal elements are the linking numbers of the orbits, i.e. the half-sum of the signed crossings of a regular projection of the orbits [9]. Since the linking number of a periodic orbit A with respect to a periodic orbit B is equal to the linking number of the periodic orbit B with respect to A, the matrix is symmetric.

Periodic orbits of period 1 and 2 are then used to identify the template describing the attractor, as shown in figure 4 (left panel). The matrix associated with the template is

$$\begin{pmatrix} 0 & +1 \\ +1 & -1 \end{pmatrix}$$

where the standard convention mentioned in [6] is used. In the main diagonal of the matrix the number of signed twistings of each branch of the template twisted around itself is shown, while the off-diagonal elements represent the number of the signed crossings of the two branches with each other. Periodic orbits of period 3–5 were also studied to confirm that this matrix is indeed that describing the topological properties of the attractor. Thus the template on the left panel of figure 4 captures all the characteristics of the basic mechanism of the proto-Lü system.

4. Comparison of the Lü and Lorenz attractors

It was pointed out in [11], for the same parameter values as in the present paper, that Lü's attractor possesses similar features with the classical Lorenz attractor. Indeed, as we saw in figure 1, the shapes of these attractors are very 'similar'. Furthermore, as shown in [12], they also share a lot of common properties: they both have three fixed points with the same stability properties, exhibit chaotic behaviour characterized by one positive Lyapunov exponent, and their routes to chaos are of the same type.

However, there exists *no diffeomorphism* transforming the Lü system to the Lorenz equations, in the sense that the two systems do not have the same set of eigenvalues at the fixed points, a property which does not hold (see, e.g., section 21 of [1], and theorem 1 of [13]). More importantly, the results of the present paper, when combined with those of [8], allow us to postulate that there does not exist a *homeomorphism* mapping the phase space of the Lü attractor to the phase space of the Lorenz attractor, transforming oriented orbits of one system to oriented orbits of the other.

This can be easily seen by comparing the templates of the two systems: the template of the proto-Lorenz system obtained after reducing (in exactly the same way as we did for the Lü system) the symmetry of the Lorenz system [8] is shown in figure 4 (right panel). The picture of this template clearly implies that the proto-Lorenz system exhibits simple horseshoe dynamics, which is not equivalent to the dynamics of the proto-Lü system, as was explained in section 3.

The difference in the underlying dynamics of these two systems is clearly seen in figure 4, as the templates differ in the way the branches named 0 and 1 are twisted around each other. This difference can also be read off from the corresponding matrices characterizing the templates. The matrix of the proto-Lorenz template

$$\begin{pmatrix} 0 & 0 \\ 0 & +1 \end{pmatrix}$$

is different from that of the Lü section (given in section 3) and this implies that these two systems, for the specific parameter values, are topologically not equivalent.

5. Covers of the proto-Lü system

Of course, in the previous sections we did not study the topology of the Lü attractor itself, but of that generated when the symmetry of the Lü system is modded out, i.e. that of the proto-Lü attractor. The relation between these two attractors is quite obvious: locally they are topologically the same, but globally they are not. Hence local topological properties proved for the proto-Lü system (3) also hold for the Lü equations (1), as well.

Lü's system is the 2-cover of the proto-Lü system, in the sense that it consists of two copies of the Lü attractor patched together in a symmetrical way. This can be easily seen in figure 5, where the projections of these two systems are shown. Lü's attractor has two symmetrically

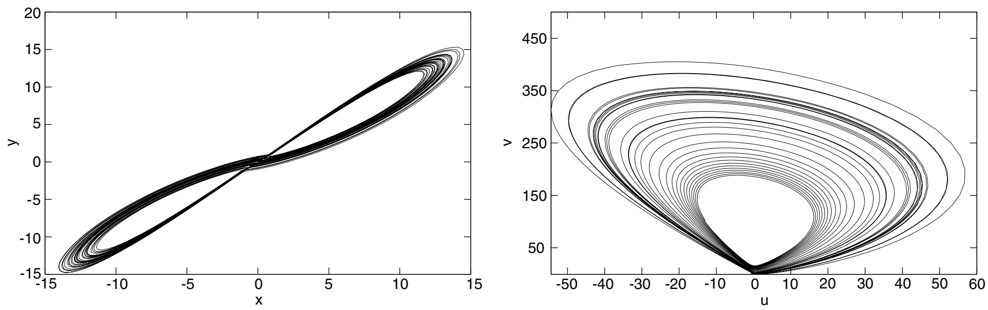

Figure 5. Projection on the x–y and u–v planes of the Lü and proto-Lü systems.

patched wings, each being locally the same with the unique wing of the proto-Lü attractor, as implied by the local diffeomorphism π introduced in section 2.

The procedure we followed to construct the proto-Lü system can, of course, be reversed, enabling us to obtain system (1) from system (3). Generalizing this procedure, one can obtain a series of symmetric attractors. Indeed, as the name implies, proto-Lü is the prototype for a family of attractors. This so-called family of n-covers of system (3) consists of attractors that are nothing more than n copies of the proto-Lü system itself, patched symmetrically together and each having n wings.

In the present section we shall follow this generalized inverting procedure to construct the 3-cover of the proto-Lü system, as described in detail in [7].

Let us view the phase space $\mathbb{R}^3$ as the cartesian product $\mathbb{C} \times \mathbb{R}$, where (u, v) represents the complex plane, under the identification $(u, v) \equiv w = u + \mathrm{i}v$, $w \in \mathbb{C}$. Note that the map

$$\begin{aligned} &\gamma_n : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R} \\ &(w, z) \mapsto (w^n, z) \end{aligned}$$

is a local diffeomorphism away from the z-axis, where n is a natural number equal to the number of copies we wish to patch together. We observe that when $n = 2$ this map is the local diffeomorphism π introduced in section 2.

If $\mathcal{L}(w, z)$ denotes the proto-Lü vector field, the following relation holds:

$$D\gamma_n(\mathcal{L}_n(w, z)) = \mathcal{L}(w^n, z)$$

where $D\gamma_n$ is the Jacobian matrix of γ_n and $\mathcal{L}_n$ denotes the n-cover of $\mathcal{L}$. To find the equations for the n-cover of the proto-Lü system we only have to solve this equation for $\mathcal{L}_n(w, z)$.

If we do this for $n = 2$, the result clearly gives system (1), since this is the 2-cover of the proto-Lü system. We therefore apply the above procedure to $n = 3$ and obtain the 3-cover of the proto-Lü system, which reads as

$$\begin{aligned} \dot{p} &= \frac{cp}{3} + \frac{1}{3M}((c+a)(q^2 - p^2) + 2pq(a - z)) - \frac{a}{3}(p - q) + \frac{qz}{3} \\ \dot{q} &= \frac{1}{3}((c - a)q - (a + z)p) + \frac{1}{3M}((a - z)(p^2 - q^2) + 2pq(a + c)) \\ \dot{z} &= \frac{3}{2}p^2 q - \frac{1}{2}q^3 - bz \end{aligned} \tag{5}$$

where M denotes the quantity $\sqrt{p^2 + q^2}$ and (p, q, z) are the new coordinates. This 3-cover attractor of the proto-Lü system is shown in figure 6. As expected, the attractor has three wings, each of them corresponding to a copy of the proto-Lü system. In a similar way,

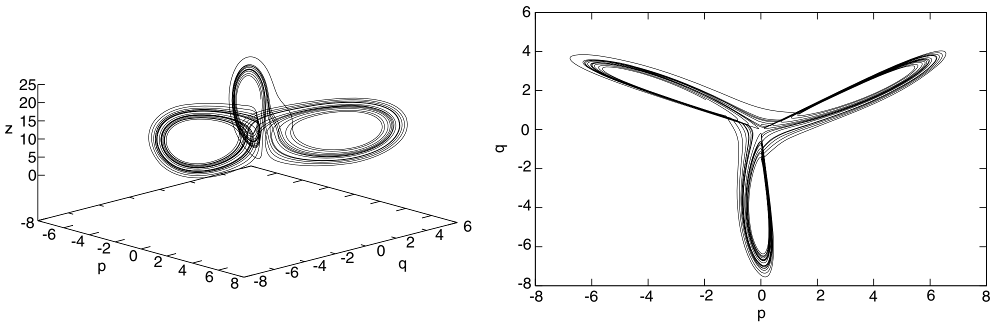

Figure 6. The 3-cover of the proto-Lü system, for $(a, b, c) = (36, 3, 13)$.

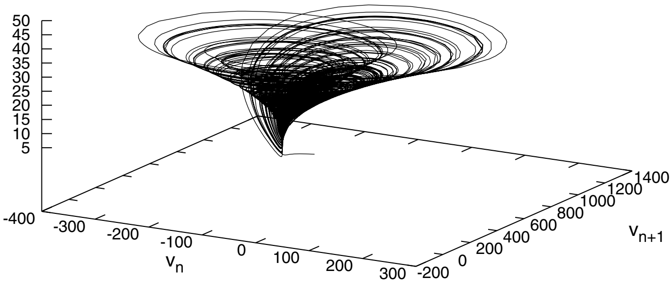

Figure 7. The attractor of the proto-Chen system, for $(a, b, c) = (35, 3, 28)$.

the 4, 5, ... etc. covers of the proto-Lü system can be constructed, leading to a series of corresponding attractors. Their topology is clear: they are just n-symmetrical copies of the mechanism described by the template in figure 4 (left panel) and this justifies the conclusion that Lü and Lorenz systems are not orbitally topologically equivalent. The Lorenz system is the double cover of the standard horseshoe mechanism, while the Lü system is not.

6. Chen's system

Chen's system is defined by the equations [10]

$$\begin{aligned}\dot{x} &= ay - ax \\ \dot{y} &= (c - a)x - xz + cy \\ \dot{z} &= xy - bz\end{aligned} \tag{6}$$

and generates a chaotic attractor, e.g. for $(a, b, c) = (35, 3, 28)$. Constructing the corresponding proto-Chen system, by the same procedure as in section 2, yields

$$\begin{aligned}\dot{u} &= (c - a)u + (2a - c)v - (c + a)N + vw \\ \dot{v} &= (c - 2a)u + (c - a)v + cN - uw - Nw \\ \dot{z} &= \tfrac{1}{2}v - bw\end{aligned} \tag{7}$$

where N stands for the quantity $\sqrt{u^2 + v^2}$. The attractor thus generated for these parameter values is shown in figure 7.

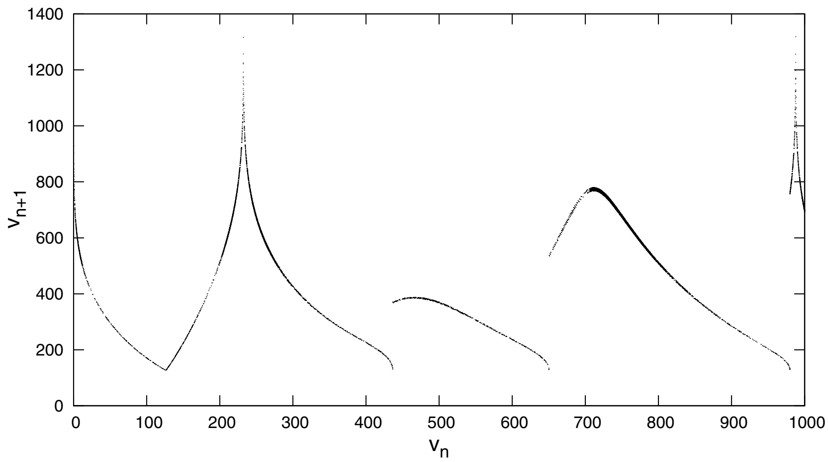

Figure 8. First return map for the proto-Chen system, when $(u, z) = (0, 21)$.

The proto-Chen system has two fixed points, as expected, since the Chen system has three: one is at the origin, and the other two are symmetric ones. The proto-Chen fixed points are (0, 0, 0) and (0, 126, 21). We therefore define the Poincaré section for this system to be

$$P = \{(u, v, w) \in \mathbb{R}^3 / w = 21\}$$

and construct the first return map with respect to the v variable. If we also fix the u value at 0, the first return map is shown in figure 8. Comparing with figure 3, we conclude that the complexity of the proto-Chen attractor is much higher than that of the proto-Lü attractor, since the proto-Chen's first return map has multiple branches, implying the existence of more branches in the template of the attractor. This fact also implies that, for the parameter values studied, Chen's attractor is topologically not equivalent to the Lorenz or Lü attractors.

Each branch of a template represents a fundamental way of knotting two different periodic orbits. We thus argue that the more freedom periodic orbits have to knot around each other the higher the complexity of the attractor. The number of branches in a template, therefore, can be used to quantify the *topological complexity* of an attractor in a way that is perhaps more fundamental than other dynamical or geometrical properties of the motion.

7. The Lorenz-like family

The fact that the Lü, Lorenz and Chen systems are not equivalent for the parameters mentioned above does not imply that they do not have common properties globally. It just emphasizes that for a deeper understanding of these systems one needs to consider a wider range of parameters.

The behaviour of a system depends crucially on its fixed points: there are three fixed points for the Lü system, namely (0, 0, 0), and the symmetry related $(\sqrt{bc}, \sqrt{bc}, c)$, $(-\sqrt{bc}, -\sqrt{bc}, c)$. Let us note that one fixed point is always located on the z-axis, while the remaining two are on the line $x = y$. Thus, the symmetry axis coincides with the double nullcline $x = y = 0$ and remains invariant under the flow, obeying the equation $\dot{z} = -bz$.

The linearized system, along the z-axis, reads

$$\begin{aligned} \dot{x} &= -ax + ay \\ \dot{y} &= -zx + cy \\ \dot{z} &= -bz \end{aligned} \tag{8}$$

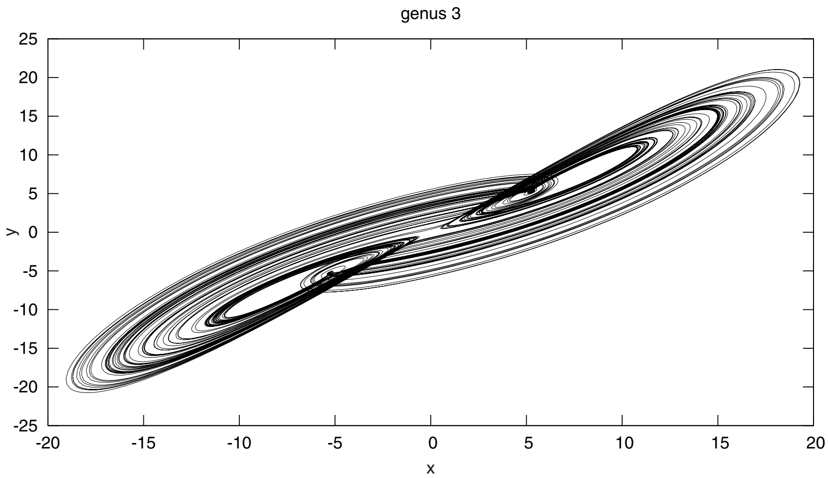

Figure 9. The Lü attractor for parameter values $(a, b, c) = (36, 3, 20)$.

allowing us to deduce that one eigenvalue $(-b)$ corresponds to the invariant z-axis while the other two $(c, -a)$ are related to the flow transverse to the symmetry axis, on the planes

$$\{(x, y, z) \in \mathbb{R}^3 / z = k\}, \qquad k \in \mathbb{R}.$$

The behaviour of the flow at different planes depends on the 'height' k of the plane.

In [15] nine dynamical systems are discussed, one of them being the classical Lorenz equations. They are all invariant under the above-mentioned symmetry and, furthermore, they all present the same qualitative properties as those of the Lü system described above. These nine systems also possess an extra characteristic: they all undergo a *perestroika*, i.e. the topological structure of the attractor they generate changes as the parameter values vary and this change is the same for all of them. The tori that bound [9] the corresponding attractors are of genus 3 for a range of parameter values, but when the parameters vary, they become tori of genus 1. These nine systems constitute the so-called Lorenz-like family [15].

The exact same perestroika, common to all members of the Lorenz-like family, occurs in the Lü attractor as well. In figure 9 the projection of the Lü attractor on the (x, y)-plane is presented for parameter values $(a, b, c) = (36, 3, 20)$. One sees from the shape of the attractor that it is bounded by a genus 3 torus. When we adjust the parameter values to $(a, b, c) = (36, 3, 28.605)$, the projection of the attractor is as shown in figure 10, making it clear that the torus bounding the attractor is of genus 1. This allows us to conclude that the Lü system also belongs to the Lorenz-like family.

It is worth noting that in paper [12] the matrix unfolding theory of Arnol'd's [3] is used to explain the similarities observed in the Lü, Lorenz and Chen systems. Specifically, it is shown there that the Lü and Chen systems are members of the universal unfolding of the Lorenz model, even at parameter values for which we have shown that the attractors are topologically not equivalent. Thus, it is clear that Arnol'd's theory of matrix unfolding cannot be invoked to justify the equivalence of dynamics, since it is possible for systems belonging to the same family to have attractors of topologically different structure. This does not contradict Arnol'd's theory since the *universal unfolding* of an object can unite in the same family objects of different topological nature.

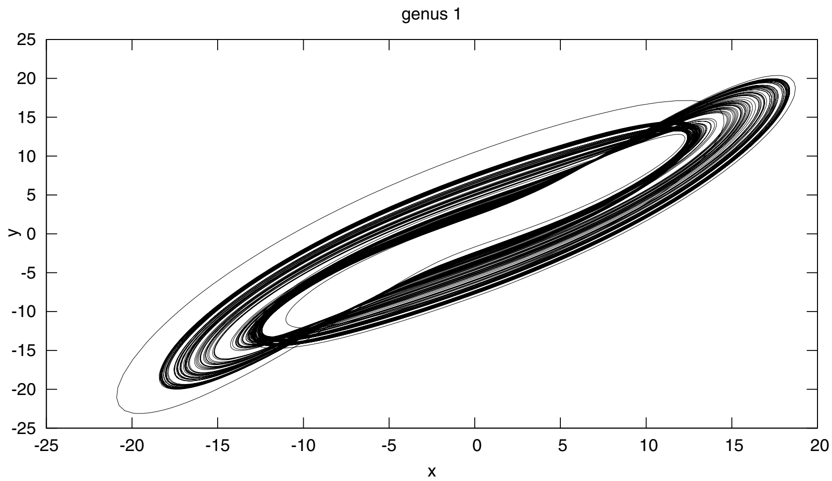

Figure 10. The Lü attractor for parameter values $(a, b, c) = (36, 3, 20)$.

8. Conclusions

In this paper we have studied the Lü system for a specific set of parameters. We first constructed a new system, called proto-Lü, obtained by modding out the symmetry exhibited by the original equations. Then, using well-established methods of knot theory, we studied the topological organization of the set of periodic orbits and extracted the template which identifies the attractor.

Combining this result with the fact that the proto-Lorenz system generates an attractor of a simple horseshoe type, we showed that these two systems are (orbitally) *not* topologically equivalent. We thus deduce that the original Lü and Lorenz attractors are not topologically equivalent, since they are just the 2-covers of their corresponding 'proto' systems.

We also constructed the 3-cover of the proto-Lü system to illustrate how one may obtain attractors with higher order symmetries. Then, we examined the attractor generated by the Chen system and found that it is considerably more complex than the Lü and Lorenz systems, according to a notion of topological complexity based on the number of different knot types that periodic orbits can produce.

The fact that two systems are topologically not equivalent, for one choice of parameter values, does not imply that they are not equivalent for other parameters. We showed that the Lü system possesses the same qualitative properties (fixed points and their stability, symmetry, nullclines, perestroikas) as all nine systems belonging to the so-called Lorenz-like family studied in [15] and therefore concluded that the Lü system should also be included in that family.

Another system which generates attractors similar to those of the Lorenz-like family is presented in [14]. The equations of that system read as

$$\begin{aligned} \dot{x} &= a(-x + y + yz) \\ \dot{y} &= y - xz \\ \dot{z} &= -bz + xy. \end{aligned} \tag{9}$$

Integrating numerically equations (9) for different values of a and b, a series of attractors is generated, some of which are pictorially similar to the classical Lorenz, Chen and Lü

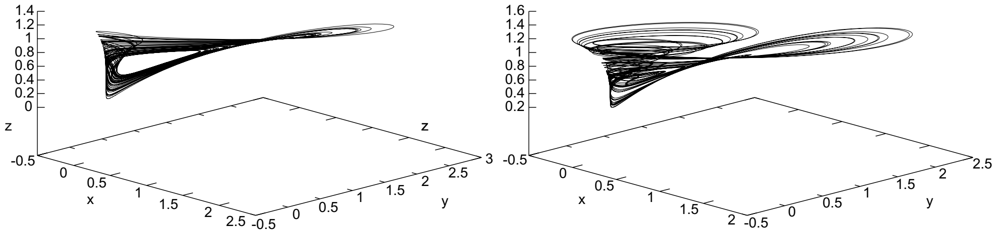

Figure 11. Attractors generated by system (9) for $(a, b) = (2.4, 0.28)$ (left) and $(a, b) = (1.5, 0.2)$ (right).

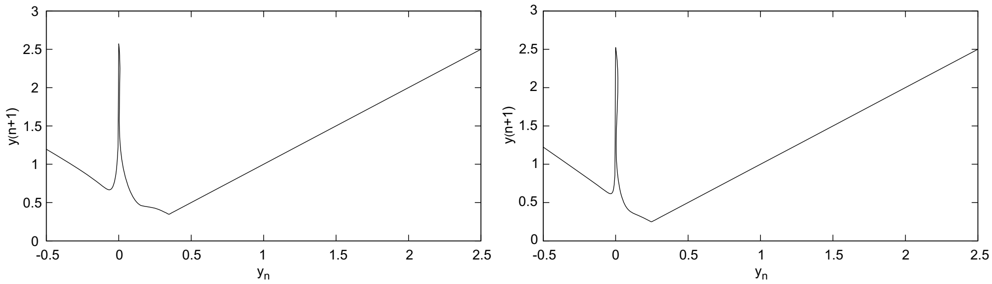

Figure 12. First return maps for system (9) when $(a, b) = (2.4, 0.28)$ (left) and $(a, b) = (1.5, 0.2)$ (right).

attractors. Constructing the 'proto' system of (9), we obtain the attractors shown in figure 11. To investigate the optical similarity of the attractor shown in figure 11 (left) with the proto-Lü attractor and the similarity of the attractor shown in figure 11 (right) with the proto-Chen attractor we construct their first return maps. The return maps, shown in figure 12 possess four branches, a fact that implies that system (9) is not similar to the systems studied in previous paragraphs. The *topological inequivalence* between system (9) and the previous two systems can, of course, be seen by the number of fixed points, as system (9) has five fixed points. As we see now, however, even the mechanism generating the strange attractors of system (9) is quite different from the corresponding mechanism of the Lorenz-like family. It would be interesting to investigate further the topological properties of system (9), for different parameter values, and compare them with analogous properties of the other systems, but that is a topic that we would like to address in a future publication.

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